Partiële differentiaalvergelijkingen, WIPDV-07 2011/12 semester II a

Examination, April 13th, 2012.

Name	**Student number**

Notes:

- You may use one sheet (single side written) with notes from the lectures.
- During the exam it is NOT permitted to consult books, handouts, other notes.
- Numerical/graphic calculators are permitted, symbolic calculators are NOT permited.
- Devices with wireless internet connection and/or document readers are NOT permitted.
- To pass the exam, You need to gather at least half of the total points at the final exam.
- Hint: please describe the solution procedures in full details, not only the results.

TEST (to be returned by 17:00)

1. **[6 pts]** Find the solution $u = u(x,t)$ of the following initial-boundary-value problem

$$u^2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0, \qquad x > 0, t > 0,$$

with

$$\begin{aligned} u(x,0) &= \sqrt{x} \quad x > 0, \\ u(0,t) &= 0 \quad\quad t > 0. \end{aligned}$$

Solution:
We use the methods of characteristics. Characteristic equations are

$$\frac{dx}{dr} = u^2$$

and

$$\frac{dt}{dr} = 1, \qquad \Rightarrow \qquad t = r.$$

Then we have

$$\frac{du}{dr} = 0 \qquad \Rightarrow \qquad u = \text{constant on characteristic } = F(x_0).$$

Hence,

$$x = u^2 r + x_0 = u^2 t + x_0,$$

since u is constant along the characteristic and dx/dr is the derivative of x along the characteristic curve. Therefore, we have

$$x_0 = x - u^2 t.$$

Thus, substituting for x_0 into $F(x_0)$ we obtain the implicit solution

$$u(x,t) = F(x - u^2 t),$$

where F is determined by the initial condition, namely, $t = 0$, $x = x_0$ and $u = \sqrt{x_0}$. Thus, $F(x_0) = \sqrt{x_0}$ and so

$$u = \sqrt{x - u^2 t}, \qquad x - u^2 t > 0.$$

Squaring both sides, we can manipulate this solution into an explicit solution for u in terms of x and t.

$$\begin{aligned} u^2 &= x - u^2 t, \\ u^2(1+t) &= x, \\ u^2 &= \frac{x}{1+t}, \end{aligned}$$

and so the final solution is

$$u = \sqrt{\frac{x}{1+t}}, \qquad x > 0, t > 0.$$

We can easily check that this is the solution to the assigned initial-boundary value problem by calculating the partial derivatives,

$$\frac{\partial u}{\partial x} = \frac{1}{2} x^{-1/2} (1+t)^{-1/2},$$

$$\frac{\partial u}{\partial t} = -\frac{1}{2} x^{1/2} (1+t)^{-3/2},$$

Therefore,

$$u^2 \frac{\partial u}{\partial x} = \frac{x}{1+t} \frac{1}{2x^{1/2}(1+t)^{3/2}} = \frac{1}{2} \frac{x^{1/2}}{(1+t)^{3/2}} = -\frac{\partial u}{\partial t}$$

2. **[6 pts]** Solve the problem

$$\begin{aligned} &u_{tt} - u_{xx} = 1, \quad && -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = x^2, && -\infty < x < \infty, \\ &u_t(x,0) = 1, && -\infty < x < \infty. \end{aligned}$$

Solution:
To obtain a homogeneous equation, we use the substitution $v(x,t) = u(x,t) - t^2/2$. The initial condition is unchanged. We conclude that v solves the problem

$$v_{tt} - v_{xx} = 0, v(x,0) = x^2, v_t(x,0) = 1.$$

Using dAlemberts formula we get

$$v(x,t) = \frac{1}{2}\left[(x+t)^2 + (x-t)^2\right] + t = x^2 + t^2 + t,$$

that is, $u(x,t) = x^2 + t + 3t^2/2$.

3. **[5 pts]** Let $u(x,t)$ be a solution of the problem

$$\begin{aligned} &u_t - u_{xx} = 0, \quad && Q_T = \{(x,t) \mid 0 < x < \pi,\ 0 < t \le T\}, \\ &u(0,t) = u(\pi,t) = 0, && 0 \leqslant t \leqslant T, \\ &u(x,0) = \sin^2(x), && 0 \leqslant x \leqslant \pi. \end{aligned}$$

Without computing the solution $u(x,t)$ explicitly, prove that $0 \leqslant u(x,t) \leqslant e^{-t}\sin(x)$ in the rectangle Q_T.

Hint: you may use the maximum principle.

Solution:
The function $w(x,t) = e^{-t}\sin x$ solves the problem

$$\begin{aligned} &w_t - w_{xx} = 0, \quad && (x,t) \in Q_T, \\ &w(0,t) = w(\pi,t) = 0, && 0 \leqslant t \leqslant T, \\ &w(x,0) = \sin(x), && 0 \leqslant x \leqslant \pi. \end{aligned}$$

On the parabolic boundary $0 \le u(x,t) \le w(x,t)$, and therefore, from the maximum principle $0 \le u(x,t) \le w(x,t)$ in the entire rectangle Q_T.

4. Consider the function given by $f(x) = |x|$ if $0 \le x \le p$.

   (a) **[2 pts]** Find the even $2p$-periodic extension, $\bar{f}(x)$, of $f(x)$ in the interval $-p \le x \le p$.

(b) **[3 pts]** Find the full Fourier series of $\bar{f}(x)$ (give also the expression for the Fourier coefficients).

(c) **[3 pts]** Judge if the Fourier series of $\bar{f}(x)$ converges pointwise to $\bar{f}(x)$ for every x.

(d) **[4 pts]** Using the Fourier series of $\bar{f}(x)$, computed at point (b), find the Fourier series of the $2p$-periodic function given by

$$g(x) = \begin{cases} a\left(1 + \frac{1}{p}x\right), & \text{if } -p \leqslant x \leqslant 0 \\ a\left(1 - \frac{1}{p}x\right), & \text{if } 0 \leqslant x \leqslant p \end{cases}$$

Solution:

(a) The $2p$-periodic extension, $\bar{f}(x)$, of $f(x)$ in the interval $-p \leq x \leq p$ is obtained by reflecting $f(x)$ with respect to the vertical axis in the interval $-p \leq x \leq 0$, and then repeating the resulting function in any other interval of length $2p$.

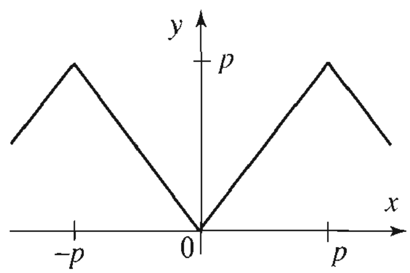

Figure. Triangular wave with period $2p$.

(b) We compute the Fourier coefficients of the full Fourier series of $\bar{f}(x)$ using the theorem seen in the course

$$a_0 = \frac{1}{2p}\int_{-p}^{p} f(x)dx = \frac{p}{2}.$$

To compute a_n we take advantage of the fact that $f(x)\cos\frac{n\pi}{p}x$ is an even function and write

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}xdx = \frac{2}{p}\int_{0}^{p} f(x)\cos\frac{n\pi}{p}xdx =$$
$$= \frac{2}{p}\int_{0}^{p} x\cos\frac{n\pi}{p}xdx = \frac{-2p}{\pi^2 n^2}\left(1 - \cos n\pi\right),$$

where the last integral is evaluated by parts. Since $\cos n\pi = (-1)^n$, $a_n = 0$ if n is even, and $a_n = \frac{-4p}{\pi^2 n^2}$ if n is odd. A similar computation shows that $b_n = 0$ for all n (since f is even). We thus obtain the Fourier series

$$f(x) = \frac{p}{2} - \frac{4p}{\pi^2}\left(\cos\frac{\pi}{p}x + \frac{1}{3^2}\cos\frac{3\pi}{p}x + \frac{1}{5^2}\cos\frac{5\pi}{p}x + \cdots\right).$$

(c) Because f is continuous and piecewise smooth, applying convergence theorems of Fourier series we may conclude that the Fourier series converge to $\bar{f}(x)$ for all x.

(d) Comparing the figure below with $\bar{f}(x)$, we see that we can obtain the graph of g by translating the graph of $\bar{f}(x)$ upward by p units, and then scaling it by a factor of $\frac{a}{p}$.

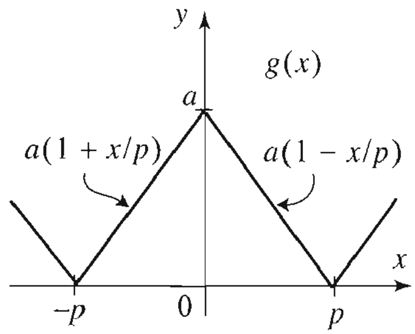

Figure A $2p$-periodic triangular wave.

This is expressed by writing

$$g(x) = \frac{a}{p}(-f(x) + p) = a - \frac{a}{p} f(x).$$

Now to get the Fourier series of g, all we have to do is perform these operations on the Fourier series of $\bar{f}$. We get

$$\begin{aligned} g(x) &= a - a\left(\frac{1}{2} - \frac{4}{\pi^2}\left(\cos\frac{\pi}{2}x + \frac{1}{3^2}\cos\frac{3\pi}{p}x + \frac{1}{5^2}\cos\frac{5\pi}{p}x + \cdots\right)\right) \\ &= \frac{a}{2} + \frac{4a}{\pi^2}\left(\cos\frac{\pi}{p}x + \frac{1}{3^2}\cos\frac{3\pi}{p}x + \frac{1}{5^2}\cos\frac{5\pi}{p}x + \cdots\right). \end{aligned}$$

In compact form, we have

$$g(x) = \frac{a}{2} + \frac{4a}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\frac{(2k+1)\pi}{p}x.$$

5. **[7 pts]** Find the solution $u(x, y)$ of the reduced Helmholtz equation $\Delta u - ku = 0$ (k is a positive parameter) in the square $0 < x, y < \pi$, where u satisfies the boundary condition

$$u(0, y) = 1, \ u(\pi, y) = u(x, 0) = u(x, \pi) = 0.$$

Solution:
We solve by the separation of variables method: $u(x, y) = X(x)Y(y)$. We obtain

$$X''Y + Y''X = kXY \Rightarrow \frac{-Y''}{Y} = \frac{X''}{X} - k = \lambda.$$

We derive for Y a SturmLiouville problem

$$Y'' + \lambda Y = 0, Y(0) = Y(\pi) = 0.$$

Therefore, the eigenvalues and eigenfunctions are

$$\lambda_n = n^2, Y_n(y) = \sin ny, n = 1, 2, \ldots$$

Then, for X we obtain

$$(X_n)'' - (k + n^2)X_n = 0 \Rightarrow X_n(x) = A_n e^{\sqrt{(k+n^2)x}} + B_n e^{-\sqrt{(k+n^2)x}}.$$

The general solution is thus

$$u(x, y) = \sum_{n=1}^{\infty} \left[A_n e^{\sqrt{(k+n^2)}x} + B_n e^{-\sqrt{(k+n^2)}x}\right] \sin ny.$$

The boundary conditions in the x direction are expressed as

$$u(0, y) = \sum_{n=1}^{\infty} (A_n + B_n) \sin ny = 1,$$
$$u(\pi, y) = \sum_{n=1}^{\infty} \left[A_n e^{\sqrt{(k+n^2)}\pi} + B_n e^{-\sqrt{(k+n^2)}\pi}\right] \sin ny = 0.$$

We expand $f(y) = 1$ into a sine series

$$1 = \sum_{n=1}^{\infty} b_n \sin ny,$$
$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin(ny) dy = \frac{-2}{\pi n} \left[(-1)^n - 1\right].$$

Comparing coefficients yields

$$A_n = -\frac{b_n e^{-\sqrt{(k+n^2)}\pi}}{e^{\sqrt{(k+n^2)}\pi} - e^{-\sqrt{(k+n^2)}\pi}}, B_n = \frac{b_n e^{\sqrt{(k+n^2)}\pi}}{e^{\sqrt{(k+n^2)}\pi} - e^{-\sqrt{(k+n^2)}\pi}}$$

6. **[4 pts]** Given a bounded region $A \subset \mathbb{R}^n$, show using Green's first identity that the Dirichlet problem

$$\Delta u(x) = f(x) \text{ when } x \in A, \quad u(x) = g(x) \text{ when } x \in \partial A$$

has at most one solution.

Solution:
If u, v are both solutions, then $w = u - v$ is harmonic and also zero on the boundary. By Green's identity then, we have

$$\int_A w\Delta w dx = -\int_A \nabla w \cdot \nabla w dx + \int_{\partial A} w \nabla w \cdot n dS,$$

where both the leftmost and the rightmost integrals are zero. This implies

$$\int_A \nabla w \cdot \nabla w dx = 0 \Longrightarrow \sum_{i=1}^{n} \int_{\partial A} w_{x_i}^2 dx = 0,$$

so w is constant. Given that $w = 0$ on the boundary, we get $w = 0$ at all points.